Quanser NI-ELVIS Trainer (QNET) Series:

QNET Experiment #03: Gantry Control

Rotary Pendulum (ROTPEN) Gantry Trainer

Student Manual

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1. Laboratory Objectives

In industry, the crane is often used to transport items from one place to another. The gantry experiment involves developing a control system for a crane travelling on a moving platform. In this case, the crane is represented by the suspended pendulum and the rotary arm behaves as the moving platform that transports the crane at different locations. The problem is to develop a controller that enables the platform, or in this case the arm of the rotary pendulum system, to track a commanded position while minimizing the motions of the crane, or pendulum, as it is being transported.

In this laboratory, the equations representing the motions of the rotary pendulum will be derived using a technique known as Lagrange. The resulting nonlinear dynamics are then then linearized and converted to a state-space model. The linear state-space representation of a system is different than Laplace and is used more prominently in the control research field. The obtained model is then used to design a closed-loop controller using the Linear-Quadratic Regulator technique. It is used to minimize or dampen the movements of the suspended pendulum while the rotary arm tracks the reference angle given. Practically speaking, the algorithm developed regulates the movements of the gantry and keeps the crane steady in the vertical down position such that items can be transported in a safely.

2. References

[1] *NI-ELVIS User Manual.*
[2] *QNET-ROTPEN User Manual.*

3. ROTPEN Plant Presentation

3.1. Component Nomenclature

As a quick nomenclature, Table 1, below, provides a list of the principal elements composing the Rotary Pendulum (ROTPEN) Trainer system. Every element is located and identified, through a unique identification (ID) number, on the ROTPEN plant represented

in Figure 1, below.

ID #	*Description*	*ID #*	*Description*
1	DC Motor	**3**	Arm
2	Motor/Arm Encoder	**4**	Pendulum

Table 1 ROTPEN Component Nomenclature

Figure 1 ROTPEN System

3.2. ROTPEN Plant Description

The QNET-ROTPEN Trainer system consists of a 24-Volt DC motor that is coupled with an encoder and is mounted vertically in the metal chamber. The L-shaped arm, or hub, is connected to the motor shaft and pivots between ±180 degrees. At the end of the arm there is a suspended pendulum attached. The pendulum angle is measured by an encoder.

4. Pre-Lab Assignments

This section must be read, understood, and performed before you go to the laboratory session.

4.1. Pre-Lab Assignment #1: Open-Loop Modeling

The ROTPEN plant is free to move in two rotary directions. That is, it has two degrees of freedom or 2 DOF. As described in Figure 2, the arm rotates about the Y0 axis and its angle is denoted by the θ while the pendulum attached to the arm rotates about another axis and its angle is called α. The pendulum angle is defined as being positive, $\alpha>0$, when rotating clockwise. Thus as the arm moves in the positive clockwise direction, the pendulum moves in the positive clockwise direction. The shaft of the DC motor is connected to the arm pivot and the input voltage of the motor is the control variable.

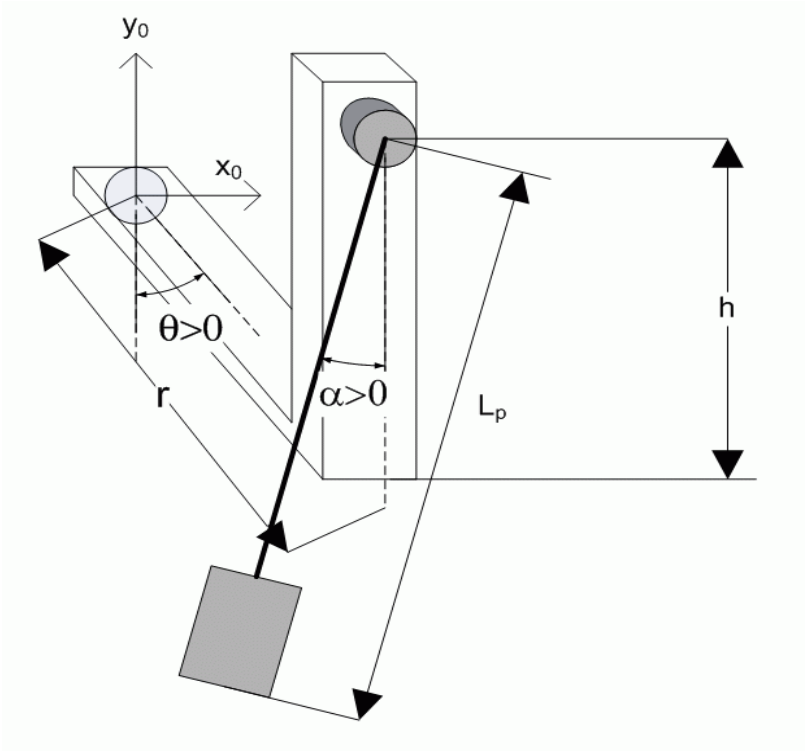


Figure 2 Rotary Pendulum System

In this pre-lab exercise, the raw nonlinear mathematical model that represents the motions of the arm and the pendulum is developed. Given nonetheless are the *nonlinear* dynamics

between the angle of the arm, θ, the angle of the pendulum, α, and the torque applied at the arm pivot, τ_{output}.

$$\frac{d^2}{dt^2}\theta(t) = -\frac{M_p{}^2\, g\, l_p{}^2\, r\cos(\theta(t))\,\alpha(t)}{(M_p\, r^2\cos(\theta(t))^2 + J_{eq})\, J_p + M_p\, l_p{}^2\, J_{eq}} + \frac{J_p\, M_p\, r^2\cos(\theta(t))\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right)^2}{(M_p\, r^2\cos(\theta(t))^2 + J_{eq})\, J_p + M_p\, l_p{}^2\, J_{eq}} + \frac{J_p\,\tau_{output} + M_p\, l_p{}^2\,\tau_{output}}{(M_p\, r^2\cos(\theta(t))^2 + J_{eq})\, J_p + M_p\, l_p{}^2\, J_{eq}} \quad [1]$$

$$\frac{d^2}{dt^2}\alpha(t) = -\frac{l_p\, M_p\,(J_{eq}\, g + M_p\, r^2\, g\cos(\theta(t))^2)\,\alpha(t)}{(M_p\, r^2\cos(\theta(t))^2 + J_{eq})\, J_p + M_p\, l_p{}^2\, J_{eq}} - \frac{l_p\, M_p\, r\sin(\theta(t))\, J_{eq}\left(\frac{d}{dt}\theta(t)\right)^2}{(M_p\, r^2\cos(\theta(t))^2 + J_{eq})\, J_p + M_p\, l_p{}^2\, J_{eq}} + \frac{l_p\, M_p\, r\,\tau_{output}\cos(\theta(t))}{(M_p\, r^2\cos(\theta(t))^2 + J_{eq})\, J_p + M_p\, l_p{}^2\, J_{eq}}$$

where the torque generated at the arm pivot from the motor voltage, V_m, is

$$\tau_{output} = \frac{K_t\left(V_m - K_m\left(\frac{d}{dt}\theta(t)\right)\right)}{R_m} \quad [2]$$

.

The ROTPEN model parameters used in [1] and [2] are defined in Table 2.

Symbol	*Description*	*Value*	*Unit*
M_p	Mass of the pendulum assembly (weight and link combined).	0.027	kg
l_p	Length of pendulum center of mass from pivot.		m
L_p	Total length of pendulum.	0.191	m
r	Length of arm pivot to pendulum pivot.	0.06668	m
J_m	Motor shaft moment of inertia.	3.00E-005	$kg{\cdot}m^2$
M_{arm}	Mass of arm.	0.028	kg
g	Gravitational acceleration constant.	9.810	m/s^2

Symbol	*Description*	*Value*	*Unit*
J_{eq}	Equivalent moment of inertia about motor shaft pivot axis.	1.23E-004	$kg \cdot m^2$
J_p	Pendulum moment of inertia about its pivot axis.		$kg \cdot m^2$
B_{eq}	Arm viscous damping.	0.000	N·m/(rad/s)
B_p	Pendulum viscous damping.	0.000	N·m/(rad/s)
R_m	Motor armature resistance.	3.30	Ω
K_t	Motor torque constant.	0.02797	N·m/A
K_m	Motor back-electromotive force constant.	0.02797	V/(rad/s)

Table 2 ROTPEN Model Nomenclature

Note that the pendulum center of mass, l_p, and the moment of inertia parameter, J_p, are not given because they will be calculated later in pre-lab exercise. Also viscous damping, which is a friction that opposes the velocity at which the structure is moving, is regarded as being negligible.

The following exercises build the rotary pendulum model incrementally. The first step is finding the kinematics of the center-of-gravity of the pendulum. In the second and third step, these are used to compute the potential and kinetic energy of the system. The fourth step introduces the principle of Lagrange and uses Euler-Lagrange equations to calculate the nonlinear equations of motion, or EOMs, that are given in [1].

4.1.1. Exercise: Kinematics

Consider the pendulum a point mass solid object. Find the forward kinematics of the pendulum center of gravity, COG, with respect to the base frame $o_0x_0y_0$. That is, express the position, x_p and y_p, and the velocity, xd_p and yd_p, of the pendulum COG in terms of the angles θ and α. The height between the pendulum pivot and the base of the arm is h = 0.127 m.

4.1.2. Exercise: Potential Energy

Express the total potential energy, $V_T(\alpha)$, of the rotary pendulum system. There is no elastic component in the system, therefore the energy that can be stored for movement is from gravity. The gravitational potential energy depends on the vertical position of the pendulum COG.

4.1.3. Exercise: Kinetic Energy

Find the total kinetic energy, T_t, of the ROTPEN system. This includes the rotational kinetic energy of the arm and the pendulum as well as the translational kinetic energy of the pendulum COG. It is reminded that the arm's equivalent moment of inertia at the pivot is J_{eq} and the inertia of the pendulum at its rotating pivot is J_p.

4.1.4. Exercise: Lagrangian of System

The *Lagrangian* of a system is

$$L = T_t - V_t \qquad [3]$$

where T_t is the total kinetic energy of the system, calculated in Exercise 4.1.3, and V_t is the total potential energy of the system, calculated in Exercise 4.1.2. The Lagrangian of a system is the difference between the total kinetic energy of the system and its total potential energy. The *Euler-Lagrange equations of motions* is a set of differential equations that describe the motions of a system and it is generated from the Lagrangian. The equations of

motion is a mathematical model that represents an actual real-world system.

Q1. Substitute the generalized coordinates q_i

$$q_1 = \theta \quad \text{and} \quad q_2 = \alpha$$

in the kinetic energy, T_t, and potential energy, V_t, expressions found in Exercise 4.1.3 and 4.1.2.

Q2. Compute the Lagrangian

$$L(q) = T_t(q) - V_t(q)$$

where

$$q = [q_1, q_2]^T.$$

That is, calculate the Lagrangian of the rotary pendulum system in terms of the q_i coordinates.

Q3. Format the solution of L(**q**) into the following quadratic structure

$$d_{11}(q_1)\left(\frac{d^2}{dt^2}q_1(t)\right) + 2\,d_{12}(q_1, q_2)\left(\frac{d}{dt}q_2(t)\right)\left(\frac{d}{dt}q_1(t)\right) + d_{22}(q_2)\left(\frac{d^2}{dt^2}q_2(t)\right) - V_t(q) = \tau_{output} \qquad [4]$$

and enter the *inertia terms* $d_{11}(q_1)$, $d_{12}(\mathbf{q})$, and $d_{22}(q_2)$, and the potential energy term, $V_t(\mathbf{q})$, into Table 3 below. Note that τ_{ouput} is the DC motor torque defined above in [2].

Inertia Term	***Expression***
$d_{11}(q_1) =$	
$d_{12}(\mathbf{q}) =$	
$d_{22}(q_2) =$	
$V_t(q_2) =$	

Table 3 Lagrangian Inertia Terms

4.1.5. Exercise: Euler-Lagrange EOM

The Euler-Lagrange equations of motion are calculated from the Lagrangian of a system using

$$\left(\frac{\partial^2}{\partial t\,\partial qdot_i}L\right)-\left(\frac{\partial}{\partial q_i}L\right)=Q_i \qquad [5]$$

where for an *n* degree-of-freedom structure $i = \{1,..,n\}$ and Q_i is called generalized force. In the rotary pendulum system, the generalized forces are

$$Q_1=\tau_{output}-B_{eq}\left(\frac{d}{dt}\theta(t)\right)$$

$$Q_2=-B_p\left(\frac{d}{dt}\alpha(t)\right).$$

However, since the viscous damping of the arm, B_{eq}, and the pendulum, B_p, are regarded as being negligible, the generalized forces become simply $Q_1 = \tau_{output}$ and $Q_2 = 0$. With that information, applying [5] to the Lagrangian expression in [4] and performing some manipulation gives

$$\begin{aligned} &d_{11}\left(\frac{d^2}{dt^2}q_1(t)\right)+d_{12}\left(\frac{d^2}{dt^2}q_2(t)\right)+c_{111}\left(\frac{d}{dt}q_1(t)\right)^2+(c_{121}+c_{112})\left(\frac{d}{dt}q_1(t)\right)\left(\frac{d}{dt}q_2(t)\right)\\ &\quad+c_{122}\left(\frac{d}{dt}q_2(t)\right)^2+\phi_1=\tau_{output}\\ &d_{12}\left(\frac{d^2}{dt^2}q_1(t)\right)+d_{22}\left(\frac{d^2}{dt^2}q_2(t)\right)+c_{211}\left(\frac{d}{dt}q_1(t)\right)^2+(c_{221}+c_{212})\left(\frac{d}{dt}q_1(t)\right)\left(\frac{d}{dt}q_2(t)\right)\\ &\quad+c_{222}\left(\frac{d}{dt}q_2(t)\right)^2+\phi_2=0 \end{aligned} \qquad [6]$$

where for $i = \{1,2\}$, $j = \{1,2\}$, and $k = \{1,2\}$, d_{ij} is the inertia terms previously calculated, c_{ijk} are called Christoffel symbols, and ϕ_k is a function of the potential energy. The Christoffel symbols are

$$c_{ijk}=\frac{1}{2}\left(\frac{\partial}{\partial q_i}d_{kj}\right)+\frac{1}{2}\left(\frac{\partial}{\partial q_j}d_{ki}\right)-\frac{1}{2}\left(\frac{\partial}{\partial q_k}d_{ij}\right) \qquad [7]$$

and

$$\phi_k(q)=\frac{\partial}{\partial q_k}V_t(q) \qquad [8]$$

.

Complete the Table 4 below by finding the Christoffel symbols using [7] and ϕ_k using [8].

Parameter	Value	Parameter	Value
$c_{111} =$		$c_{112} =$	
$c_{121} =$		$c_{212} =$	
$c_{211} =$		$c_{212} =$	
$c_{221} =$		$c_{222} =$	
$\phi_1 =$		$\phi_2 =$	

Table 4 Lagrangian Christoffel symbols and ϕ_k

4.1.6. Exercise: Euler-Lagrange Matrix Form

The equations of motion of the rotary pendulum have now been found. The matrix form of the Euler-Lagrangian equations of motion in [6] is commonly seen as

$$\mathrm{D}(q)\left(\frac{d^2}{dt^2}\mathrm{q}(t)\right) + \mathrm{C}\left(q, \frac{d}{dt}\mathrm{q}(t)\right)\left(\frac{d}{dt}\mathrm{q}(t)\right) + \mathrm{g}(\mathrm{q}(t)) = \tau \qquad [9]$$

where for a 2 DOF system

$$\mathrm{D}(q) = \begin{bmatrix} d_{11}(q_1) & d_{21}(q) \\ d_{21}(q) & d_{22}(q_2) \end{bmatrix},$$

$$\mathrm{g}(\mathrm{q}(t)) = [\phi_1(\mathrm{q}(t)), \phi_2(\mathrm{q}(t))]^T,$$

$$\tau = [\tau_{output}, 0]^T,$$

and C(**q**,**q_dot**) is the matrix that includes the Christoffel symbols in Table 4. Substitute the expression calculated from d_{ij} in Table 3 along with c_{ijk} and ϕ_k in Table 4 into the Euler-Lagrange expression in [6] and map it to the matrix form shown in [9]. Fill out the matrices below in Table 5.

Matrix	Value
$\mathrm{D}(\mathrm{q}(t))$	
$\mathrm{C}\left(\mathrm{q}(t), \frac{d}{dt}\mathrm{q}(t)\right)$	
$\mathrm{g}(\mathrm{q}(t))$	

Table 5 Euler-Lagrangian Matrices

4.2. Pre-Lab Assignment #2: Finding the Linear State-Space Model

The general state model of a time-invariant linear continuous-time dynamical system is

$$\frac{d}{dt}\mathrm{x}(t) = A\,\mathrm{x}(t) + B\,\mathrm{u}(x)$$
$$\mathrm{y}(t) = C\,\mathrm{x}(t) + \mathrm{D}\,\mathrm{u}(x) \qquad [10]$$

where **x**(t) is the *n*-dimensional state, and for an *r*-input and *m*-output system $\mathrm{A}\in\Re^{n\times n}$, $\mathrm{B}\in\Re^{n\times r}$, $\mathrm{C}\in\Re^{m\times n}$, and $\mathrm{D}\in\Re^{m\times r}$ are constant matrices. The goal of this section is to find the *linear state-space* model of the rotary pendulum system. In particular, the goal is to calculate the *A* and *B* matrices. Before going through the actual exercise, an overview of the process of attaining the model will be given.

The mathematical model of the rotary pendulum found in [9] is *nonlinear*. Therefore [9] must be linearized about a operating point in order to fit the system into the state-space from shown in [10].

4.2.1. Exercise: Linearizing EOM

Consider a continuous, nonlinear two-variable function *f*(x,y) that is to be linearized about the operating point (x_o,y_o). The linearization is calculated as follows

$$f_l(x, y) = \mathrm{f}(x_o, y_o) + \left(\frac{\partial}{\partial x_o}\mathrm{f}(x_o, y_o)\right)(x - x_o) + \left(\frac{\partial}{\partial y_o}\mathrm{f}(x_o, y_o)\right)(y - y_o)$$

.

The linearized function f_l(x,y) is an approximation of the nonlinear function f(x,y) and this approximation only holds in a region about the operating point. That is, $f_l(x,y) \approx f(x,y)$ when $(x-x_o,y-y_o) \in C$, where C is a two-dimensional region about the origin.

The linearized Euler-Lagrange equation can be given by

$$\mathrm{D}_l(q)\left(\frac{d^2}{dt^2}\mathrm{q}(t)\right) + C_l\left(q, \frac{d}{dt}\mathrm{q}(t)\right)\left(\frac{d}{dt}\mathrm{q}(t)\right) + g_l(q) = \tau \qquad [11]$$

where D_l(**q**) and C_l(**q**,**q_dot**) are the linearized inertia and Christoffel matrices and g_l(**q**) is the linearized g(**q**). In the case of the ROTPEN system, the linearized mathematical model represents the motions of the real physical system only about the point where the angles were linearized.

Linearize each element of the D(**q**) and C(**q**,**q_dot**) matrices as well as the vector g(**q**) about the operating point

$$x_o = [\,0,\ 0,\ 0,\ 0\,]^T$$

and record the results in Table 6.

Matrix	***Value***
$\mathrm{D}_l(\mathrm{q}(t))$	
$C_l\left(\mathrm{q}(t), \frac{d}{dt}\mathrm{q}(t)\right)$	
$g_l(\mathrm{q}(t))$	

Table 6 Linearized Lagrangian Matrices

4.2.2. Exercise: Solving for EOM Acceleration Terms

The system has now been linearized and is more easily manageable. The linearized Euler-Lagrange equations of motions in [11] must be solved for **q_ddot**, such that

$$\frac{d^2}{dt^2}\mathrm{q}(t) = -\mathrm{D}_l(q)^{[-1]}\, C_l\left(q, \frac{d}{dt}\mathrm{q}(t)\right)\left(\frac{d}{dt}\mathrm{q}(t)\right) - \mathrm{D}_l(q)^{[-1]}\, g_l(q) + \mathrm{D}_l(q)^{[-1]}\,\tau \quad [12]$$

.

Calculate the solution of [12] from the answer found in Exercise 4.2.1 and structure the resulting vector in the following two-equation form

$$\frac{\partial^2}{\partial t^2} q_1 = a_1\, q_2 + b_1\, \tau_{output}$$
$$\frac{\partial^2}{\partial t^2} q_2 = a_2\, q_2 + b_2\, \tau_{output} \quad [13]$$

,

where a_1, a_2, b_1, and b_2 are all real number constants. Record the parameters in Table 7.

Solution Parameter	Expression
a1	
a2	
b1	
b2	

Table 7 Linearized Lagrange Solution Parameters

4.2.3. Exercise: Finding State-Space Model

At this point the nonlinear Euler-Lagrange equations of motions have been linearized about the origin and solved for the acceleration of the angles *q1_ddot* and *q2_ddot*. It is now ready to be placed in the state-space form.

Define the state

$$x = [x_1, x_2, x_3, x_4]^T \qquad [14]$$

as

$$\mathrm{x}(t) := \left[q_1(t), q_2(t), \frac{d}{dt} q_1(t), \frac{d}{dt} q_2(t) \right]^T .$$

Recall the generalized coordinates definitions used for the Lagrange: $q_1(t)=\theta(t)$ and $q_2(t) = \alpha(t)$. The state defined with respect to the rotary pendulum angles are

$$x_1 = \theta \text{ , } x_2 = \alpha \text{ , } x_3 = \frac{\partial}{\partial t}\theta \text{ , and } x_4 = \frac{\partial}{\partial t}\alpha .$$

Substitute the state defined in [14] into the Lagrange solution computed in [13] and place the answer in the single-input linear state-space representation

$$\frac{d}{dt}\mathrm{x}(t) = A\,\mathrm{x}(t) + B\,\mathrm{u}(x) \qquad [15]$$

where *A* is a real 4×4 matrix, *B* is a real 4×1 constant matrix, and the control input is the torque being applied by the motor, $\mathrm{u}(\mathbf{x}) = \tau_{output}(\mathbf{x})$. Enter the resulting state-space matrices in Table 8.

State-Space Matrix	Expression
A	
B	
C	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
D	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Table 8 Linear State-Space Matrices

The 4×4 matrix C and the 4×1 matrix D in the state model equation

$$y(t) = C\,x(t) + D\,u(x)$$

are given in Table 8. It is assumed from the rank of *C* that all states are being measured by sensors. Although in actuality the position angles θ and α are measured by encoders and the speed of these angles is calculated using an observer. The observer differentiates the angle and removes any high-frequency components using a low-pass filter. Thus from a controls

point of view, the output equation should be $y = [x_1,x_2]^T$ and there should be an observer that estimates the states x_3 and x_4. The controller designed later would then be in the form $u = -Kx_{est}$ and not $u = -Kx$.

4.2.4. Exercise: Adding Actuator Dynamics

The state equations developed in the last exercise are relative to the torque being applied at the motor shaft. The torque, however, is not controlled by the computer directly and is, instead, a result of the voltage being given to the DC motor. The torque generated at the arm pivot from the motor voltage, V_m, is

$$\tau_{output} = \frac{K_t\left(V_m - K_m\left(\frac{d}{dt}\theta(t)\right)\right)}{R_m}.$$

Re-define the state-space equation [15] in terms of $u(\mathbf{x}) = V_m$ and write the new matrices in Table 9.

State-Space Matrix	Expression
A	
B	

Table 9 State-Space Matrices in terms of V_m

4.3. Pre-Lab Assignment #3: Calculating the Inertia and Center of Mass of the Pendulum

The free-body diagram of the pendulum used in the QNET rotary pendulum system is shown in Figure 3.

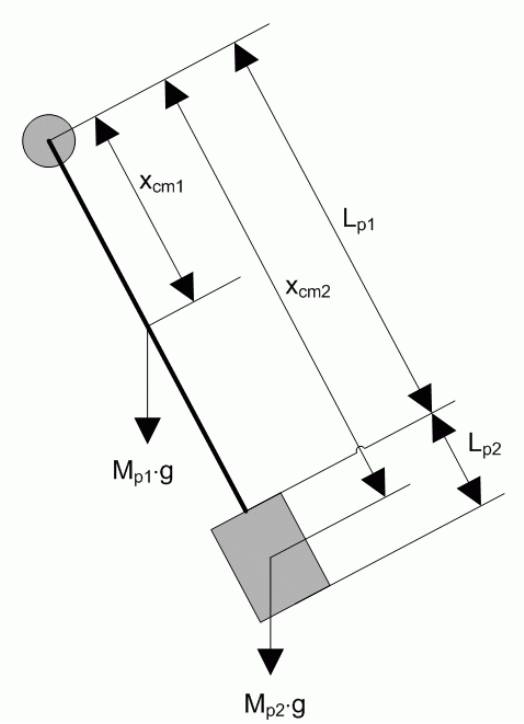


Figure 3 Pendulum Free-Body Diagram

The circle in the top-right corner represents the axis of rotation that goes into the page. The pendulum assembly is a rigid body that is composed of two bodies: the pendulum link, M_{p1}, and the pendulum weight, M_{p2}, located at the end of the link. The mass and length parameters depicted in Figure 3 that are needed for this exercise are given below in Table 10.

Symbol	***Description***	***Value***	***Unit***
M_{p1}	Mass of the pendulum link.	0.008	kg
M_{p2}	Mass of the pendulum weight.	0.019	kg
L_{p1}	Length of pendulum link.	0.171	m
L_{p2}	Length of pendulum weight.	0.019	m
l_p	Length of pendulum center of mass from pivot.		m
J_p	Pendulum moment of inertia about its pivot axis.		$kg \cdot m^2$

Table 10 Pendulum Parameters for Exercise 4.3

The first exercise is calculating the center of mass of the pendulum assembly, l_p, and the second exercise is computing the moment of inertia of the pendulum system, J_p.

4.3.1. Exercise: Calculating Center of Mass

The center of mass of a rigid body in the form of a beam can be calculated using

$$x_{cm} = \frac{\int p\, x\, dx}{\int p\, dx} \qquad [16]$$

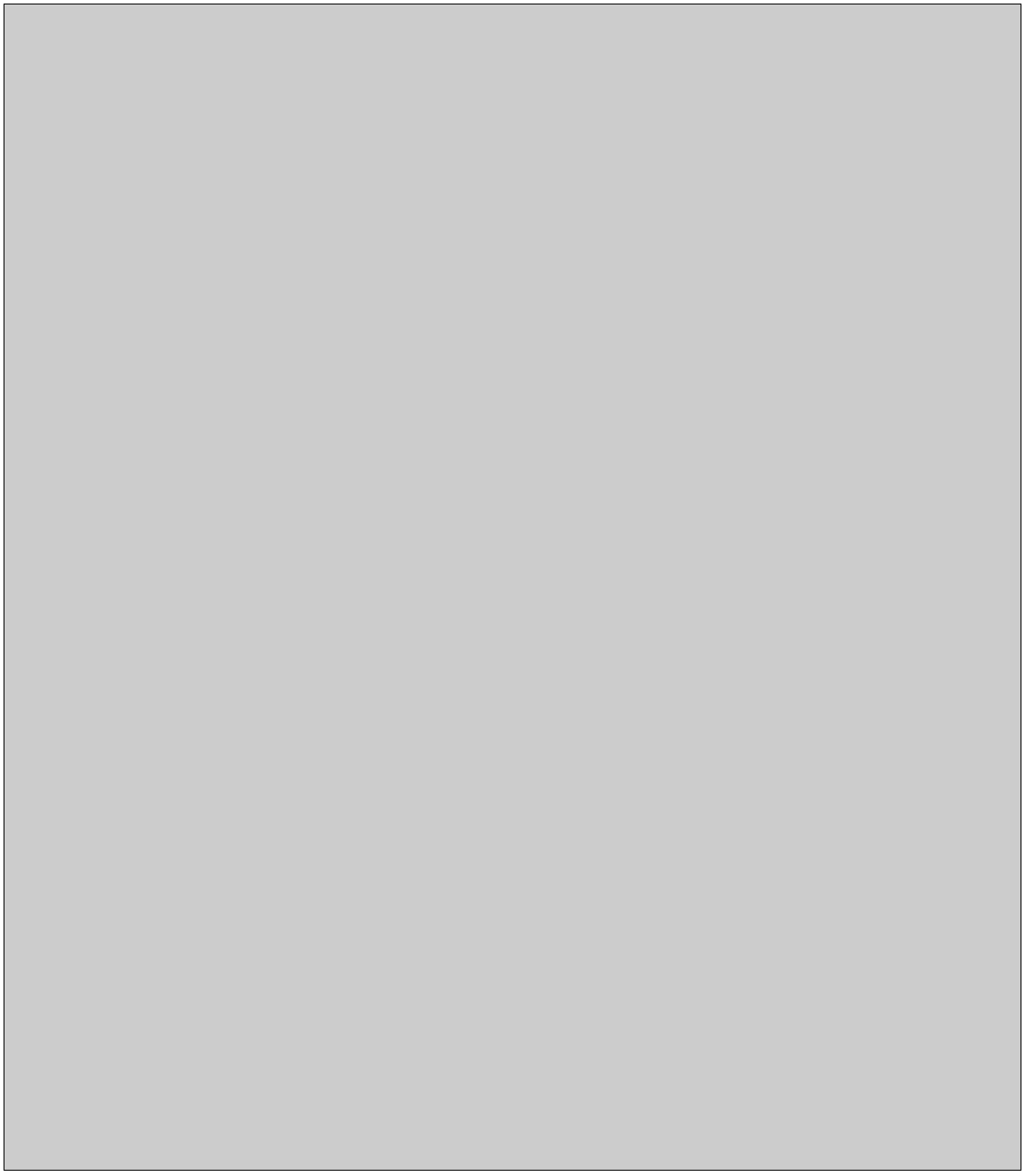
where p is the density of the body.

Express the center of mass of the pendulum link, x_{cm1}, and the pendulum weight, x_{cm2}, in terms of L_{p1}, L_{p2}, M_{p1}, and M_{p2}. These variables are all shown in Figure 3. The uniform density of the link is

$$p_1 = \frac{M_{p1}}{L_{p1}}$$

and the uniform density of the pendulum weight is

$$p_2 = \frac{M_{p2}}{L_{p2}}.$$



The pendulum is composed of two different bodies each with their own CM. The center of mass of the body as a whole can be calculated and is useful when the pendulum is to be considered as a single rigid object. The center of mass of a composite object that contains *n*

bodies can be calculated using

$$x_{cm} = \frac{\sum_i x_i m_i}{\sum_i m_i} \qquad [17]$$

where x_i is the known center of mass of body i and m_i is the mass of body i.

If the ROTPEN is considered a single rigid body, its CM would be as shown in Figure 4 and not as two separate bodies in Figure 3. The total mass of the pendulum assembly is M_p and the total length is L_p, both are quantified in Table 2.

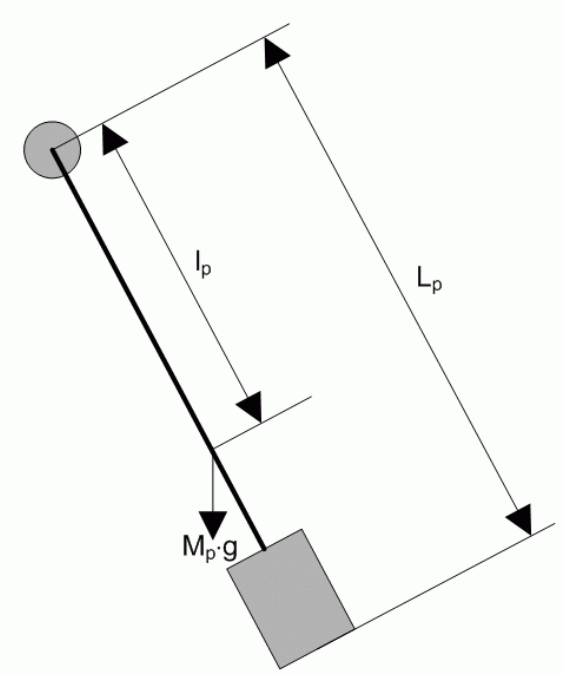


Figure 4 Free body diagram of pendulum considered a single rigid body.

1. Express the center of mass of the pendulum assembly in terms of the CM of the pendulum link, x_{cm1}, and the CM of the pendulum mass, x_{cm2}, using expression [17].
2. Calculate the numerical answer using the mass and length parameters given in Table 10 and enter the resulting center of mass, l_p, in Table 10. **Record the answer for later use in *QNET Laboratory #4 – ROTPEN Inverted Pendulum*.**

4.3.2. Exercise: Calculating Moment of Inertia

The moment of inertia J of a rigid body is expressed as

$$J = \int r^2 \, dm \qquad [18]$$

where r is the perpendicular distance between the element mass, dm, and the axis of rotation.

The moment of inertia at the pivot axis of the pendulum is an important parameter for the gantry experiment because it indicates the tendency of the pendulum, which represents a crane, to continue swinging. Thus a pendulum with more inertia is more difficult to control then one with less tendency to continue rotating when the arm ceases to move.

1. Express the moment of inertia of the pendulum J_p in terms of the pendulum link mass, M_{p1}, and its length, L_{p1}, along with the mass of the pendulum weight, M_{p2}, and the weight's length, L_{p2}, using general expression [18] and Figure 3. ***Hint***: Recall from Exercise 4.3.1 that the pendulum is a composite of two thin rods with the uniform densities $p_1 = M_{p1}/L_{p1}$ and $p_2 = M_{p2}/L_{p2}$. The integral can be evaluated by changing the mass element to the distance differential $dm = p*dr$, where p is the linear density.
2. Compute the pendulum moment of inertia and enter the result in Table 10. The mass and length of the pendulum link and the pendulum weight are specified in Table 10. **Record the answer for later use in *QNET Laboratory #4 – ROTPEN Inverted Pendulum*.**

4.4. Control Design

The goal of this section is to introduce the notion of controllability of a system and the idea behind the Linear-Quadratic Regular, or LQR, control technique. This is an introduction to designing a linear controller in the state-space.

4.4.1. Controllability

Consider this arbitrary system

$$\frac{d}{dt}x_1(t) = x_2$$
$$\frac{d}{dt}x_2(t) = u \quad . \qquad [19]$$

If there exists a control input *u* that can drive a state **x** to a vector **v** in a finite time of $t_1>0$, then vector **v** is said to be reachable. For example, Figure 5 is a phase plot of system [19] when the system begins at an initial state $x(0) = [1,1]^T$. The origin is said to be a reachable because there is an input that can steer the state from $[1,1]^T$ to $[0,0]^T$ in a finite time of t_1. In precise terms, v = [0,0] is a reachable vector because there exists a control input *u* such that $x(0) = [1,1]^T$ can be driven to $x(t_1) = [0,0]^T$.

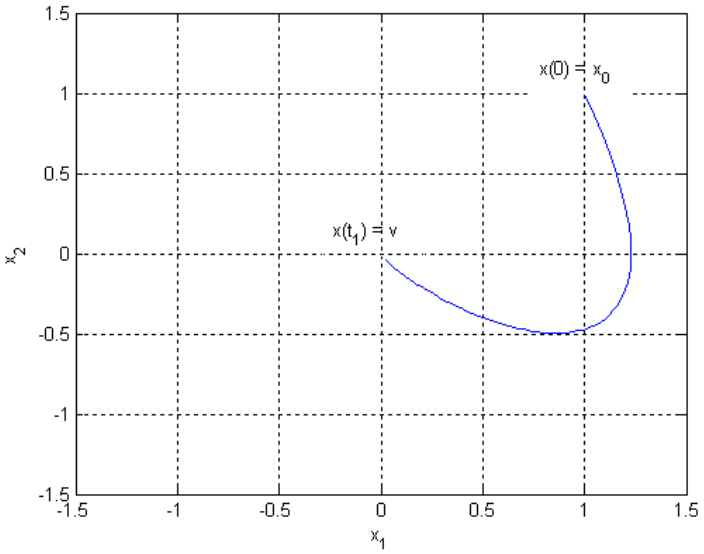


Figure 5 Phase Plot of a System

If every state is reachable then (A,B) is said to be a *controllable pair*, or the system is controllable. A system is controllable if an only if

$$rank[B\ AB\ A^2B...\ A^{(n-1)}B] = n \qquad [20]$$

where *n* is the order of the system (i.e. the number of states).

For example, applying the controllability check on system [19] gives

$$\text{rank}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 2 .$$

This implies that the system is controllable because the rank is equivalent to the number of states. Therefore a controller entering input *u* can be designed such that a given initial state converges to any target state. The *linear quadratic regulator*, or LQR, is one control design method that can be used to accomplish this and will now be introduced.

4.4.2. Linear Quadratic Regulator

The *linear quadratic regulator* problem is: given a plant model

$$\frac{d}{dt}\mathrm{x}(t) = A\,\mathrm{x}(t) + B\,\mathrm{u}(t) \qquad [21]$$

find a control input *u* that minimizes the cost function

$$J = \int_0^\infty \mathrm{x}(t)^T\, Q\, \mathrm{x}(t) + \mathrm{u}(t)^T\, R\, \mathrm{u}(t)\, dt \qquad [22]$$

where Q is an $n \times n$ positive semidefinite weighing matrix and R is an $r \times r$ positive definite symmetric matrix. That is, find a control gain *K* in the state feedback control law

$$u = K\,x \qquad [23]$$

such that the quadratic cost function *J* is minimized. The control law in [23] is known as the *optimal control law* because it finds a unique solution to the LQR problem. Note that LQR control assumes the state is fully known (i.e. no observers can be used), implying that there would be a sensor for each state.

Figure 6 gives a typical closed-loop control system. The *Q* and *R* matrices are set by the user and that effects the optimal control gain that is generated to minimize *J*. The closed-loop control performance is effected by changing the *Q* and *R* weighing matrices.

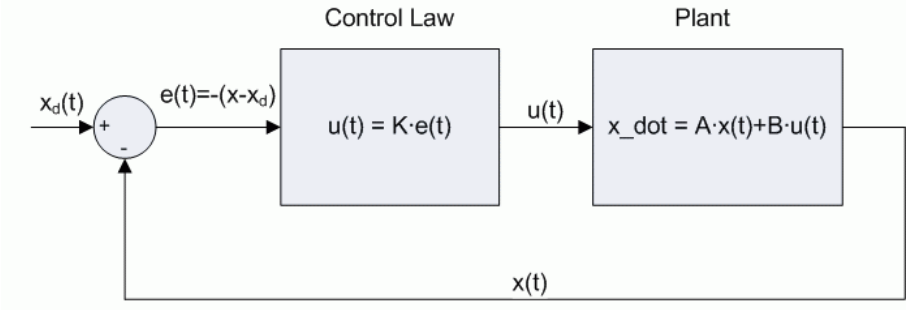


Figure 6 Closed-Loop Control System

For example, assume the plant model in [19] represents the movement of a linear cart where state x_1 is the position of the cart and x_2 is the speed of the cart. For the reference command state

$$x_d = [\,x_{d,\,1},\, 0\,]^T$$

the controller takes the form

$$u = -k_p\,(x_1 - x_{d,\,1}) - k_v\,x_2$$

where the optimal control gain is

$$K = [k_p, k_v]$$.

This is a PV controller with a proportional gain, k_p, and a velocity gain, k_v that makes the cart track a commanded position $x_{d,1}(t)$. Table 11 lists the resulting PV gains generated using the linear quadratic regular method from different weighing matrices.

Case	*Q*	*R*	*K*
1	$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$R = 1$	$K = [1.0000, 1.7321]$
2	$Q = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$	$R = 1$	$K = [2.2361, 2.3393]$
3	$Q = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$	$R = 1$	$K = [1.0000, 2.6458]$

Table 11 Resulting LQR gain from different Q and R matrices

The first case may not have resulted in overall suitable tracking, therefore some weight is placed on the top-left *Q* matrix element to place emphasis on the proportional control. This translates to the LQR algorithm working harder on the x_1^2 term to minimize the *J* cost function and it generates a larger k_p. However, as shown in Table 11, it was necessary to increase both k_p and k_v gains to minimize *J* due to the inherent system dynamics. Perhaps the k_p gain generated in case 1 provided suitable tracking performance except that the closed-loop response tends to overshoot too much. The overshoot can be dampened by augmenting the velocity gain. The bottom-right element of *Q* is increased in case 3 and results in a larger k_v while maintaining the proportional gain steady.

The *R* weighing matrix is kept at 1 in all cases for comparison purposes. If *R* is decreased this forces the control input *u* to work harder to minimize *J*, resulting in an overall higher gain *K*.

4.4.3. Gantry Control Specifications

The whole point of developing a linear model of the gantry and calculating its various parameters are so a gain can be generated that will control the gantry. In the end, it all comes down to tuning the *Q* weighing matrix, as in the Section 4.4.2, such that a closed-

loop response meets certain requirements. The LQR control design specifications are:

(1) **Tracking**: $\theta(t)$ track commanded angle $\theta_d(t)$ with $t_p<1.2$ s and $t_s < 2.3$ s.
(2) **Dampening**: $|\alpha(t)| < 7.5$ and $t_s < 6$ s
(3) **Control input limit**: $V_m < 5$ V
where t_p is the peak time, t_s is time for the 1% settling time, and V_m is the motor input voltage.

These are similar specifications to a crane moving boxes within a factory floor. The pendulum link may be thought of as the actual crane and the weight at the end represents the box to be moved. The arm is the device moving the crane. Thus the arm must track a given commanded position and do that with reasonable speed (for productivity purposes), thus low peak time. This task must be performed accurately as well to place the box in the correct location, thus low steady-state error with a reasonable settling time is a requirement. However, the arm control alone does not guarantee that the box will be delivered to the target location accurately, or safely, because the crane holding the box (i.e. the pendulum and the pendulum weight at the end) is prone to large swings. The swings are minimized by having a control that takes into account the position and speed of the crane's angle. Lastly, the controller must meet the performance specifications within the voltage capability of the DC motor.

5. In-Lab Session

5.1. System Hardware Configuration

This in-lab session is performed using the NI-ELVIS system equipped with a QNET-ROTPEN board and the Quanser Virtual Instrument (VI) controller file *QNET_ROTPEN_Lab_03_Gantry_Control.vi*. Please refer to Reference [2] for the setup and wiring information required to carry out the present control laboratory. Reference [2] also provides the specifications and a description of the main components composing your system.

Before beginning the lab session, ensure the system is configured as follows:

QNET Rotary Pendulum Control Trainer module is connected to the ELVIS.
ELVIS Communication Switch is set to BYPASS.
DC power supply is connected to the QNET Rotary Pendulum Control Trainer module.
The 4 LEDs +B, +15V, -15V, +5V on the QNET module should be ON.

5.2. Experimental Procedure

Please follow the steps described below:

Step 1. Read through Section 5.1 and go through the setup guide in Reference [2].
Step 2. Run the VI controller *QNET_ROTPEN_Lab_03_Gantry_Control.vi* shown in Figure 7.

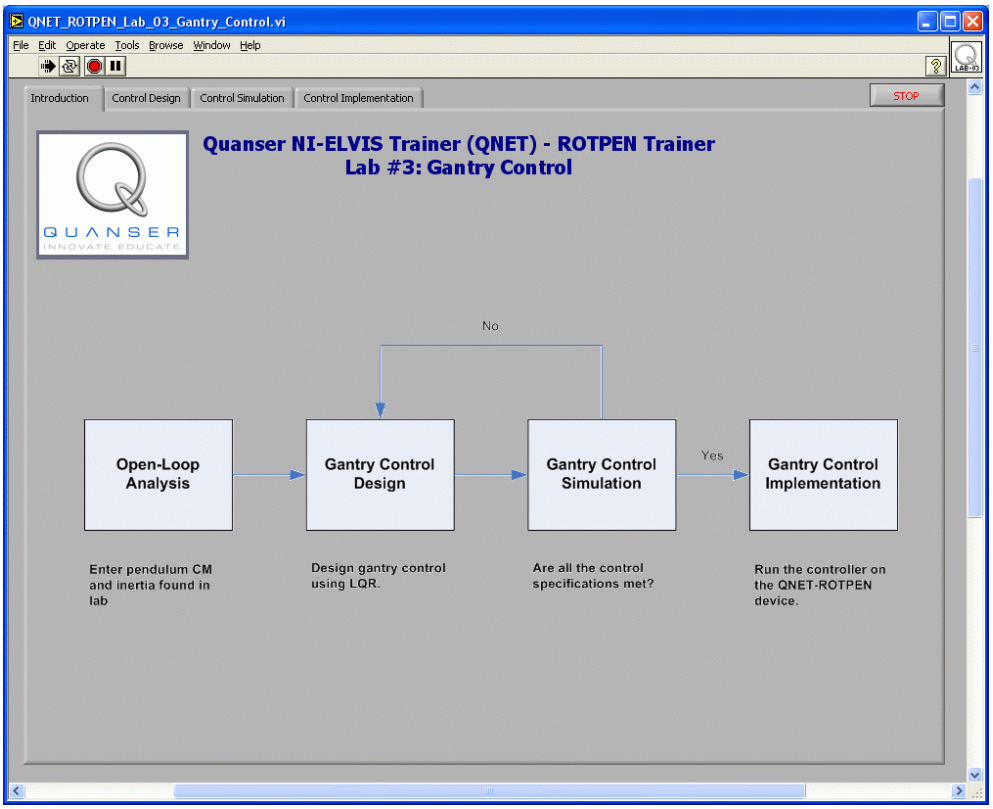


Figure 7 QNET-ROTPEN VI

Step 3. Select the the *Control Design* tab shown in Figure 8.

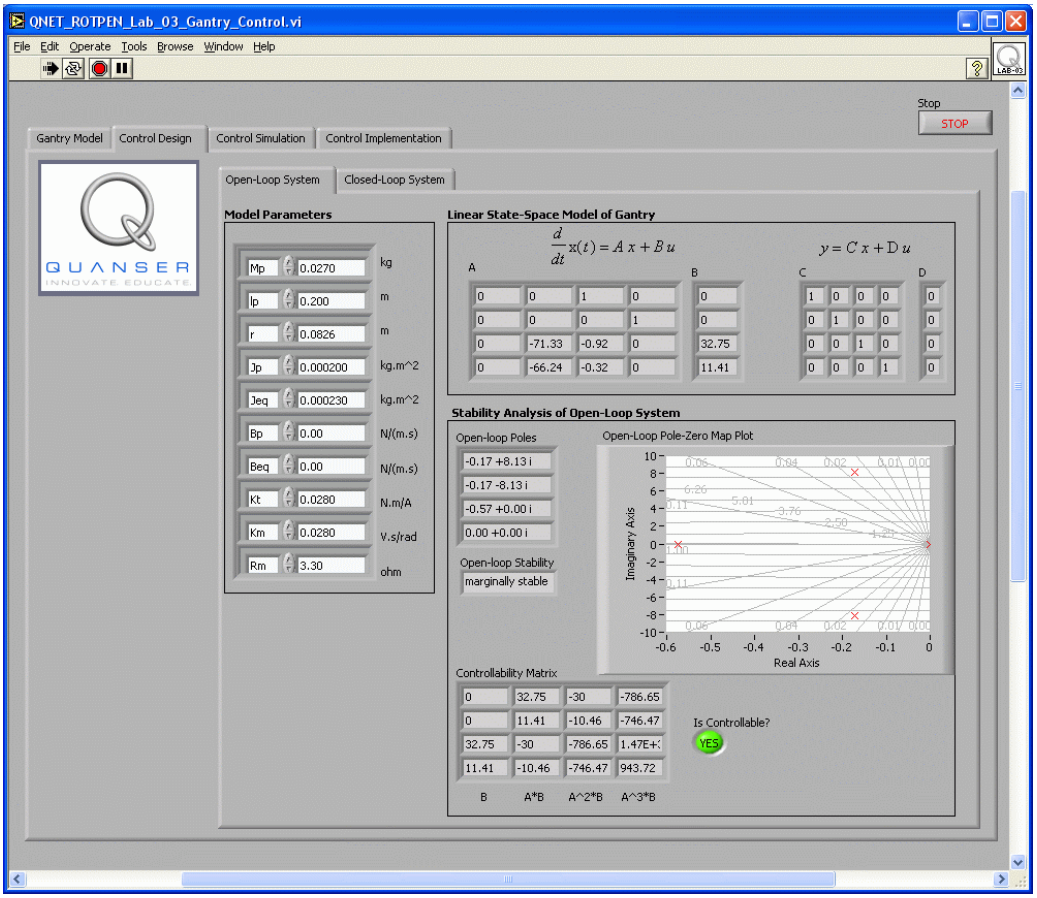


Figure 8 Open-Loop Stability Analysis

Step 4. Update the model parameter values in the top-right corner with the pendulum center of mass, l_p, and the pendulum's inertia, J_p, that both calculated in Exercise 4.3 and entered in Table 10. The linear-state space model matrices A and B, on the top-right corner of the front panel, as well as the open-loop poles, situated directly below the state matrices, are automatically updated as the parameters are changed. Vary the inertia of the pendulum, J_p, as indicated and observe the changes in the locations of the poles.

Jp (kg.m^2)	***Poles***			
	1	***2***	***3***	***4***
1.00E-004				
2.00E-004				

Jp (kg.m^2)	*Poles*			
	1	*2*	*3*	*4*
4.00E-004				
8.00E-004				

Step 5. How does increasing the inertia effect the open-loop poles and the stability of the gantry? Shortly explain how a pendulum with more inertia results in this trend.

Step 6. Under the open-loop poles in this VI it indicates the stability of the gantry system as being *marginally stable*, as shown in Figure 8. According to the poles, why is the open-loop gantry considered to be *marginally stable* and not *stable*?

Step 7. As depicted in Figure 8, the controllability matrix is shown in the bottom-right area of the front panel along with an LED indicating whether the system is controllable or not. The rank test of the controllability matrix verifies that the gantry is a controllable system, that is

$$rank[B\ AB\ A^2B\ A3B] = 4$$

is equal to the number of states in the system. Since the system is controllable, a controller can be developed. Click on the *Closed-Loop System* tab shown in Figure 9 to begin the LQR control design.

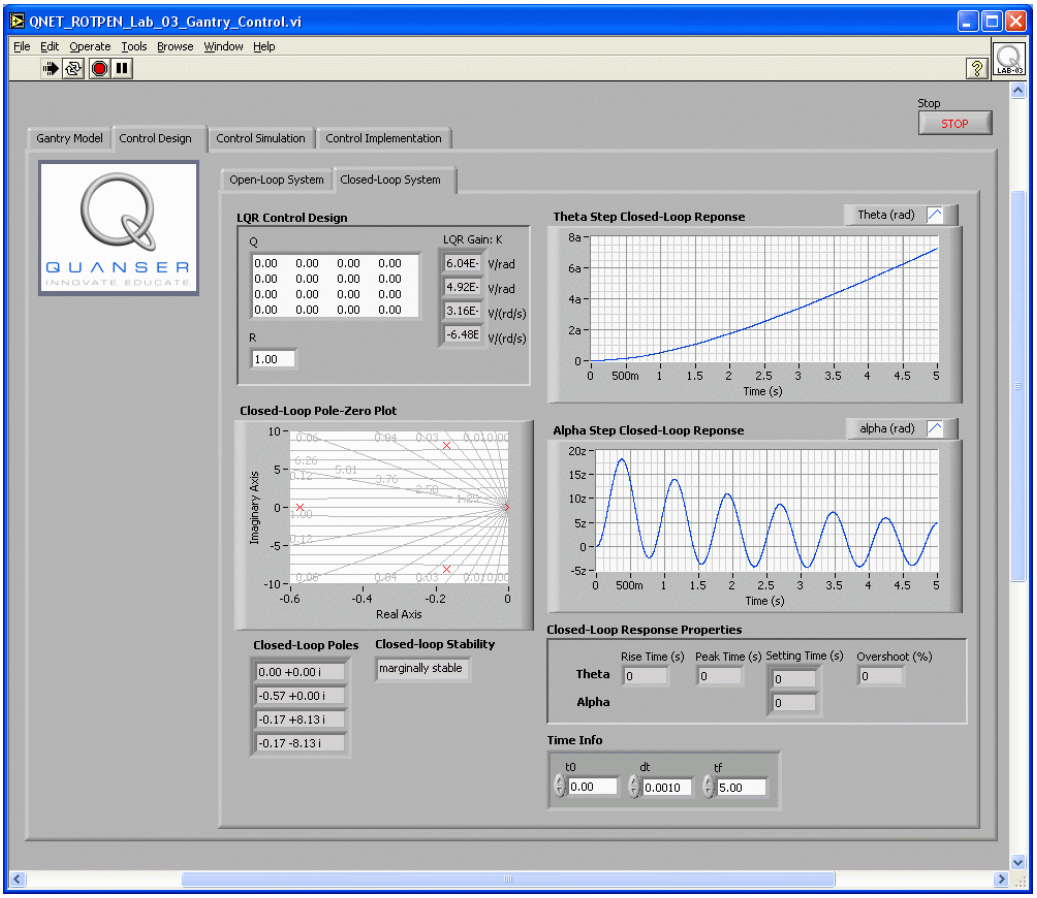


Figure 9 LQR Control Design Front Panel

Step 8. The *Q* and *R* weighing matrices and the resulting control gain *K* is in the top-left corner of the panel. Directly below the *LQR Control Design* section is a pole-zero plot that shows the locations of the closed-loop poles. The numerical value of the poles are given below the plot along with the resulting stability of the closed-loop system. The step response of the arm angle, θ(t), and the pendulum angle, α(t), are plotted in the two graphs on the right side of the VI, as shown in Figure 9. The rise time, peak time, settling time, and overshoot of the arm response and the settling time of the pendulum angle response is given. Further, the start time, duration, and final time of these responses can be changed in the *Time Info* section located at the bottom-right corner of the VI.

Step 9. For the *Q* and *R* weighing matrices

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}$$

$$R = 1$$

vary the q_1, q_2, q_3, and q_4 elements as specified in Table 12 and enter the obtained time-domain characteristics in the same table. Referring to the feedback loop in Figure 6, for the LQR gain

$$K = [k_{p,\theta}, k_{p,\alpha}, k_{v,\theta}, k_{v,\alpha}]^T \quad [25]$$

the control input *u(t)* that enters the DC motor input voltage is

$$V_m = k_{p,\theta}(x_1 - x_{1,d}) + k_{p,\alpha} x_2 + k_{v,\theta} x_3 + k_{v,\alpha} x_4 \quad [26]$$

where $k_{p,\theta}$ is the proportional gain acting on the arm, $k_{p,\alpha}$ is the proportional gain of the pendulum angle, $k_{v,\theta}$ is the velocity gain of the arm, and $k_{v,\alpha}$ is the velocity gain of the pendulum. Observe the effects that changing the weighing matrix *Q* has on the gain *K* generated and, hence, how that effects the properties of the closed-loop step response. See Section 4.4.2 for some guidance on tuning an LQR controller.

Q				*θ*			*α*
q1	***q2***	***q3***	***q4***	***t_p (s)***	***t_s (s)***	***OS%***	***t_s (s)***
5	0	0	0				
5	0	0	5				
5	0	0.5	5				

Table 12 LQR Control Design

Step 10. Re-stating the LQR specifications given in Section 4.4.3:

(1) **Tracking**: $\theta(t)$ track commanded angle $\theta_d(t)$ with t_p<1.2 s and t_s < 2.3 s.

(2) **Dampening**: $|\alpha(t)|$ < 7.5 and t_s < 6 s

(3) **Control input limit**: V_m < 5 V

where t_p is the peak time, t_s is time for the 1% settling time, and V_m is the motor input voltage. Find the q_1, q_2, q_3, and q_4 elements that results in specifications

(1) and (2) being satisfied. When the response meets requiremetns (1) and (2) move on to the next step to test the third specification.

Step 11. Although the specifications are met, there is guarantee that the control input, the motor voltage V_m, is not going out of its range. Control design is often limited by the actuator. Through simulation, it can be checked whether the control signal is going beyond 5 Volts. Select the *Control Simulation* tab and the front panel illustrated in Figure 10 should load.

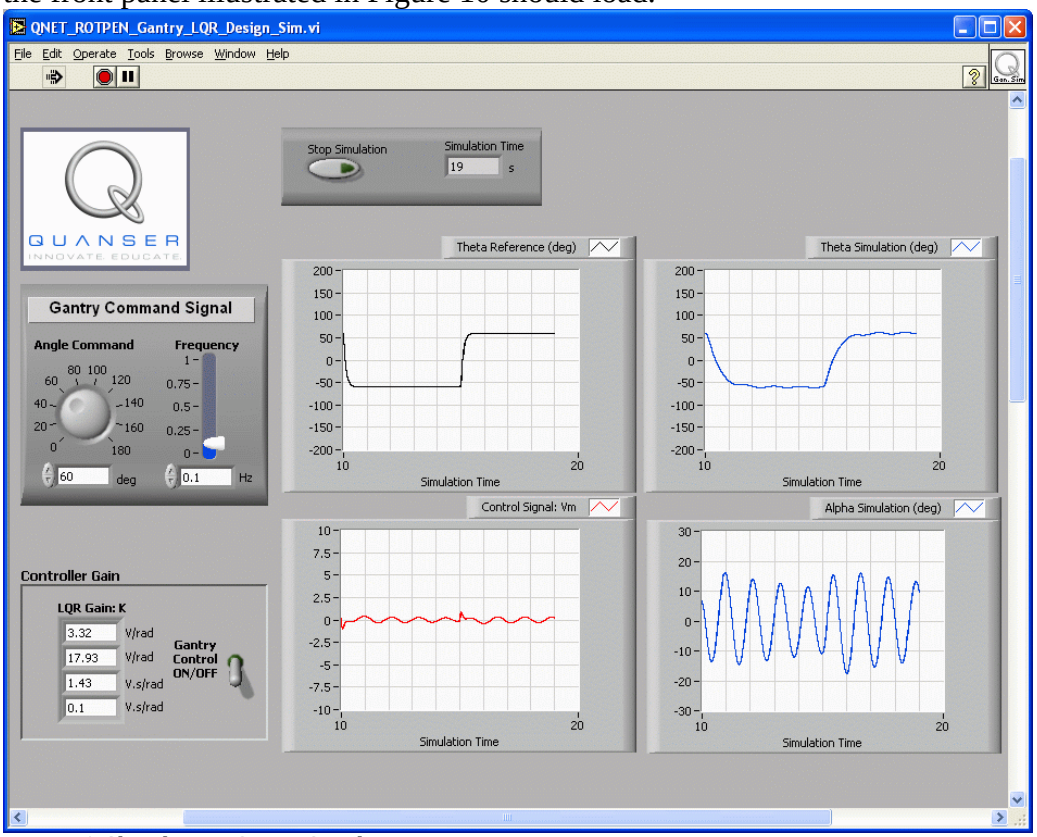


Figure 10 Closed-Loop Gantry Simulation VI

Step 12. The *Gantry Command Signal* panel enables the user to vary the amplitude and frequency of the smoothed square signal. The position command signal, denoted $\theta_d(t)$, is plotted on the top-left graph in Figure 10. The LQR gain generated by the LQR design in the previous step is shown below in the bottom-left corner along with a switch that activates the gantry control. The OFF or down position only enables the arm tracking but does nothing to dampen the pendulum. For example, as shown in Figure 10 the closed-loop $\theta(t)$ simulation on the top-right is more-or-less tracking the reference signal shown in the top-left plot. However, the bottom-right plot shows the pendulum angle

$\alpha(t)$ and it is swinging back and forth over ±10°. When the switch is activated the gantry control implemented previously is simulated and that dampens the swinging of the pendulum. Lastly, the DC motor input voltage is simulated on the bottom-left scope. Experiment by switching the GANTRY Control switch ON and OFF and changing the reference signal.

Step 13. For an angle command of 120 degrees at 0.1 Hertz, verify that the gantry control signal V_m does not exceed ±5V specification. Make sure the GANTRY control switch is ON when observing control signal. If V_m satisfies requirement (3) click on the *Acquire Data* button to return to the *Control Design* VI and go to the next step. If V_m does not meet specification (3) and went over the limit, click on the *Acquire Data* button, which bring you back to the control design VI, and tune your controller such that gain K is decreased. Return to the simulation, by clicking on the *Control Simulation* tab, with the re-tuned controller and confirm that the control input does not exceed ±5V.

Step 14. At this point a control has been found that satisfies specifications (1), (2), and (3). Enter Q matrix elements q_1, q_2, q_3, and q_4 in the last row of Table 12 along with the resulting peak time, settling time, and overshoot for θ as well as the settling time for α. The control is now ready to be implemented on the actual gantry device. Click on the *Control Implementation* tab and the VI shown in Figure 11 should open.

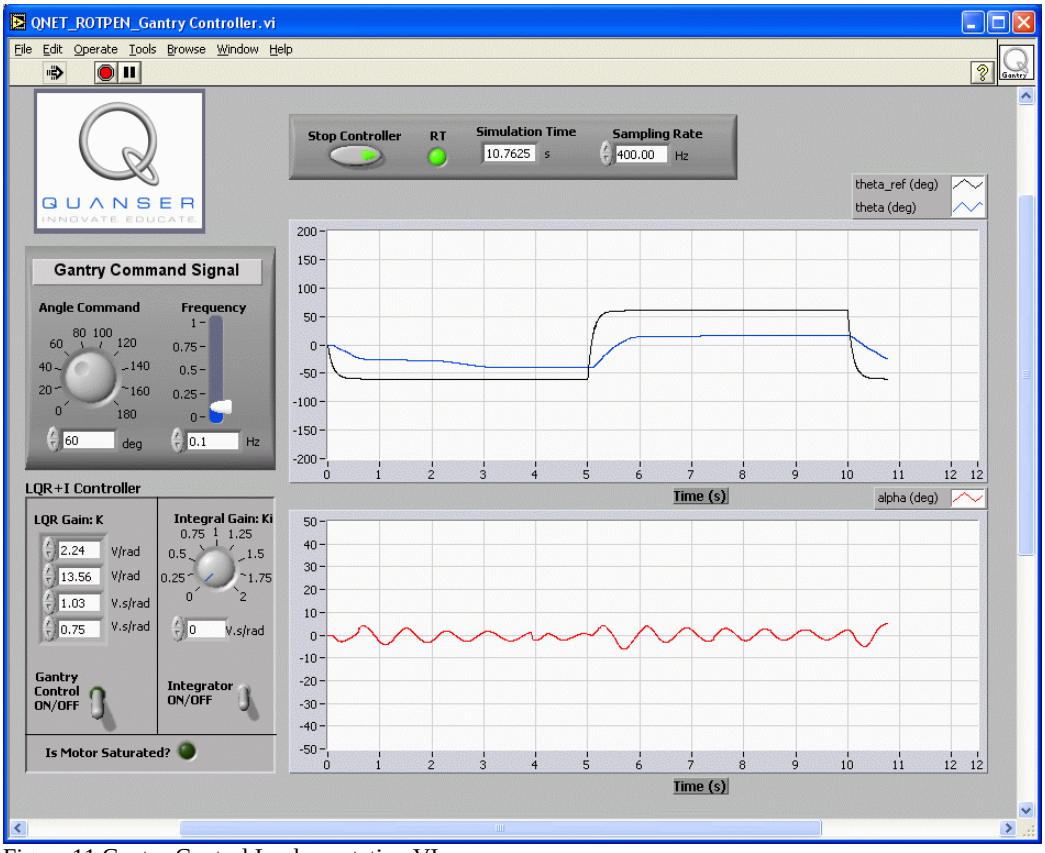


Figure 11 Gantry Control Implementation VI

Given that the QNET-ROTPEN system has been powered properly, the arm should be rotating back and forth. Similarly to the control simulation VI, the command position is a smoothed square signal that can be controlled through the *Gantry Command Signal* panel on the top-left of the VI, shown in Figure 11. The reference signal is plotted in the top graph along with the actual angle of the arm measured by encoder. The bottom scope plots the pendulum angle. By default when the VI opens, the *GANTRY control* is turned OFF (down position) and the *Integrator control*, which will be explained later, is also turned OFF. Below the control gains is an LED that warns the user if the control voltage is saturating the motor.

If the *Is Motor Saturated?* LED goes ON, click on the *Stop Controller* button in the top panel immediately!

The *Stop Controller* button stops the control and returns the user to the control design tab where adjustments to the control can be made or the session can be ended. Also in the top panel shown in Figure 11 is the RT LED that indicates if real-time is being held, the simulation time readout, and the sampling rate.

Slow down the sampling rate if the RT LED is either RED or flickering between GREEN and RED. Also, stop the control by clicking on the *Stop Controller* button return to the *Control Implementation* tab for the new sampling rate to take effect.

Step 15. Set the command angle to 120° and the frequency of the reference signal to 0.1 Hz. The pendulum should be swinging in excess of ±10° as seen on the $\alpha(t)$ plot and visually on the actual device. Keeping in mind the behaviour of the pendulum when there is no gantry control, set the *Gantry Control* switch to ON to activate the full LQR control designed. This should dampen the pendulum angle and meet specification (2). The pendulum should remain suspended in the downward vertical position as the arm swivels to track the commanded position. Thus the arm should more-or-less track the reference signal and the pendulum should remain about its suspended 0 degree position.

Activated Controller		***Arm Tracking***			***Pendulum Dampening***	
Gantry Control	K_i (V/rad/s)	t_p (s)	t_s (s)	e_{ss} (deg)	t_s (s)	***max*** $\lvert\alpha\rvert$ (deg)
OFF	0					
ON	0					
ON						

Table 13 Closed-loop time response characteristics

Step 16. As probably noticed, the implemented gantry control does not perform as well as the simulated control. Namely, there is a large steady-state error when θ track θ_d and small oscillations about $\alpha = 0°$ are not being dampened. Why are these phenomenon seen only when implementing the controller?

Step 17. The steady-state error can be significantly reduced by adding an integral component. Thus adding an integrator with gain K_i to the LQR control loop shown in Figure 6 gives the closed-loop system depicted in Figure 12. Thus the control becomes

$$V_m(t) = u_{lqr}(t) + u_{int}(t) \quad [27]$$

where the LQR control and integral control are

$u_{lqr}(t) = -K(\mathrm{x}(t) - x_d(t))$ and

$$u_{int}(t) = -\frac{K_i(x_1(t) - x_{1,d}(t))}{s} \quad [27]$$

.

The integrator control basically pumps more voltage into the DC motor to eliminate the offset between the actual arm position and the commanded arm position.

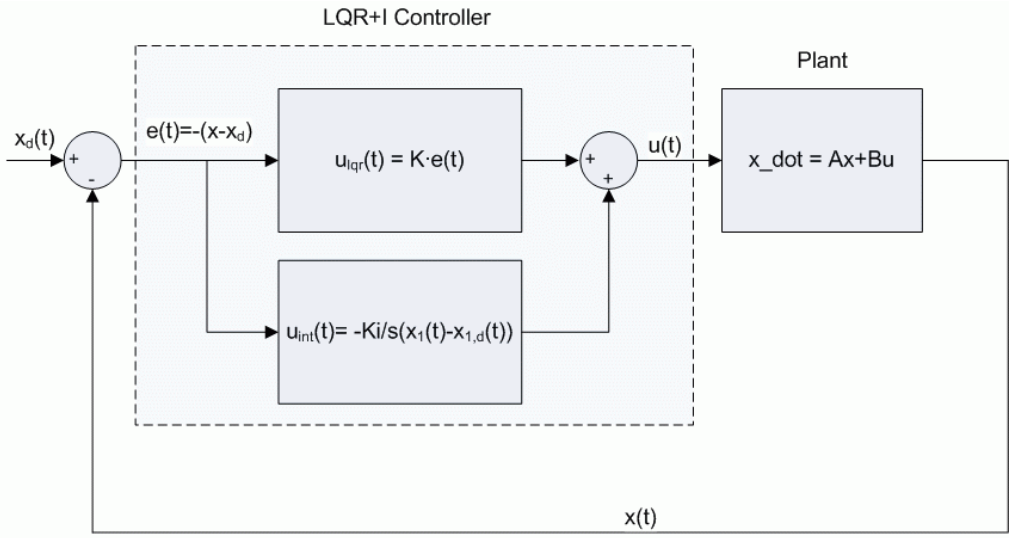


Figure 12 LQR+I Closed-Loop System

Step 18. The task now is to tune the integrator gain K_i until $\theta(t)$ converges to $\theta_d(t)$. Thus the control parameter will be tuned as the controller is being ran on the device. Record the K_i gain used in Table 13 along with the corresponding response properties.

If the *Is Motor Saturated?* LED goes ON, click on the *Stop Controller* button in the top panel immediately! In this case, reduce the K_i until the LED goes OFF because the gain is set too high and is saturating the motor.

Step 19. Click on *Stop Controller* and the *Control Design* tab should be selected. If all the data necessary to fill the shaded regions of the tables is collected, end the *QNET-ROTPEN Gantry* laboratory by turning off the *PROTOTYPING POWER BOARD* switch and the *SYSTEM POWER* switch at the back of the ELVIS unit. Unplug the module AC cord. Finally, end the laboratory session by selecting the *Stop* button on the VI.